Kazhdan-Lusztig conjecture via quasimaps

KL conjecture $=$ Thm by (Beilinson-Bernstein
Brylinski-Kashiwara, Kashiwara-Tanisaki

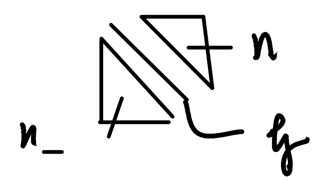
D-modules $\nearrow$

algebraic $\nearrow$ Elias-Williamson

$\mathfrak{g}$: cpx simple Lie algebra (or affine Lie algebra)

$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$

e.g. $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$



G : corresponding Lie group $\curvearrowright$ flag variety G/B $\quad \mathrm{Lie}\, B = \mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$

e.g. $\{0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n \mid \dim E_i = i\}$

representation theory of $\mathfrak{g}$ or G $\longleftrightarrow$ geometry of G/B

e.g. Borel-Weil theory

irreducible finite dim'l rep $= H^0(\mathcal{L})$

$\mathcal{L}$: line bdle over G/B

<u>Ex.</u> Weyl character formula. Suppose λ : dominant integral wt $\in \mathfrak{h}^*$
i.e. $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall i$

$$\Longrightarrow \quad \mathrm{ch}\, L(\lambda) = \frac{1}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} \sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}$$

ρ = Weyl vector $= \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha$

or $\rho(h_i) = 1$ $\forall i$

- geometric proof by fixed pt formula

$G/B \leftarrow T$ $\quad$ Lie $T = \mathfrak{h}$ $\quad$ $(G/B)^T \cong W$ (Weyl grp) $\quad$ e.g. $W = \mathfrak{S}_r$

subspaces spanned by coord. vectors

$$\sum (-1)^i \dim H^i(\mathcal{L}) = \sum_{w\in(G/B)^T = W} (\text{local contribution}) \leftarrow \frac{\mathrm{ch}(\mathcal{L}_w)}{\mathrm{ch}(\Lambda_{-1} T^*_w(G/B))}$$

KL conjecture is about ∞-dim'l irreducible representations of $\mathfrak{g}$.

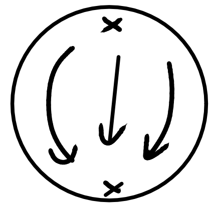
$L(\lambda)$ $\quad$ $\lambda \in \mathfrak{h}^*$ not nec. int. dominant

continuous parameter

Answer is given interms of intersection cohomology cpx of Schubert varieties.

What are Schubert varieties?

$$G/B = \coprod X_w$$



$$X_w = \{x \mid \lim_{t\to 0} t\cdot x = w\}$$

$$\dim F_\bullet \cap F_\bullet^0 \geq \cdots$$

$IC(\overline{X_w})$ at y

reflect singularities of $\overline{X_w}$ at pt y

Proof uses D-modules on G/B

($\mathfrak{g} \to$ vector fields on G/B

$U(\mathfrak{g}) \to$ diff. operators

◎ work in progress with Braverman + Finkelberg

A different geometric realization of $U(\mathfrak{g})$ via quasimaps

$Q^\alpha = \{$ based quasimaps $f: \mathbb{P}^1 \longrightarrow {}^L G/{}^L B \}$

Advantage of our approach

1) more general $\quad \mathfrak{g} \rightsquigarrow$ other $\quad$ affine Lie alg.

variants $\quad$ W-algebras. $\quad U_q(L\mathfrak{g})$, ...

2) uniform in λ

construct "universal" representation $M(\lambda)$

λ : variable in $\mathfrak{h}^*$

$\mathbb{C}^* \curvearrowright \mathbb{P}^1$, $T \curvearrowright G/B \quad \Longrightarrow \quad \mathbb{C}^* \times T \curvearrowright Q^\alpha$

$(\mathrm{Lie}\,{}^LT)^*$

$\rotatebox{90}{$\in$}$

<u>Th</u> [Braverman] (based on FFKM, ...)

$U({}^L\mathfrak{g}) \curvearrowright \bigoplus_\alpha IH^*_{\mathbb{C}^*\times T, c}(Q^\alpha)$

Langlands dual

highest weight $= \lambda$

$\hbar$: deformation parameter

$U_\hbar(\mathfrak{g}) \xrightarrow{\hbar=1} U(\mathfrak{g})$

$\xrightarrow{\hbar=0} S(\mathfrak{g})$

character formula can be derived from a general machinery of convolution algebras, developed by Ginzburg:

$A \subset \mathbb{C}^* \times T$ subtorus $\quad (Q^\alpha)^A$

e.g. $\quad \parallel$

$\{(t, \lambda(t))\} \qquad \lambda: \mathbb{C}^* \to T$

$$\Longrightarrow (Q^\alpha)^{\mathbb{C}^*} = \{ f: \mathbb{P}^1 \to G/B,\ \mathbb{C}^*\text{-equiv. map},\ \infty \mapsto e \}$$

$$\cong \{ f(1) \in G/B \} \cong X_w \cap X^e$$

$f(0) = w \in G/B$

$f(\infty) = e$